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Elastic field for a straight dislocation in a decagonal quasicrystal

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Abstract. A straight dislocation line in a decagonal quasicrystal is considered. For a decagonal quasicrystal of point groups 10, $\overline{10}$ and $10/m$ with a straight dislocation parallel to the periodic direction, the basic governing equations are solved by decoupling this problem into a plane elasticity problem and an antiplane elasticity problem. For the latter, the solution is known; for the former, it is reduced to a single equation, $\Delta^4 F(x, y) = 0$, Δ being the harmonic operator, by introducing a displacement function F. A general solution is formulated. Using a general solution and the Fourier transform, the explicit expressions for the dislocation-induced elastic field in a decagonal quasicrystal are obtained and the energy per unit length on the dislocation is given.

1. Introduction

Quasicrystals—solids with a long-range orientational order and a long-range quasiperiodic translational order [1, 2]—have become the topic of considerable experimental and theoretical studies in physics of condensed matter. Structural, electronic, magnetic, thermal and mechanical properties of quasicrystals has been investigated intensively. In particular, the research of defects has attracted extensive attention in both experiment and theory. For conventional crystals, the elasticity theory of defects, as outlined in [3] and [4], has been established and developed over 40 years. For quasicrystals, however, within the framework of the Landau theory, the elasticity theory of defects such as dislocations was formulated in 1985 [2, 5, 6], although the first experimental evidence for the existence of dislocations in quasicrystals was not provided until 1989 [7–9]. According to the continuum theory of dislocations in the general scheme of quasicrystal elasticity theory [10], the explicit expressions for the elastic field, in particular for the displacement field, induced by a dislocation have been obtained in several quasicrystals [11–15]. The first work in this area may be that of De and Pelcovits [16], who analysed a dislocation in a planar pentagonal quasicrystal and gave the analytical expressions for dislocation-induced displacement field by the iterative approach. So far many methods such as the Green function method [11], the Eshelby method [14] and the displacement function method [15] have been developed to derive analytical expressions for dislocation-induced elastic field in various quasicrystals.

This paper considers a straight dislocation line in a decagonal quasicrystal with Laue class $10/m$, or with point groups 10, $\overline{10}$ and $10/m$, where the dislocation line is parallel to the periodic direction. A general solution is suggested by introducing a displacement function for a planar decagonal quasicrystal with point group 10 in section 2. The analytical expressions for

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the displacement field as well as stress field induced by a dislocation are obtained in section 3. The elastic field and the energy per unit length on the straight dislocation line are both given in section 4.

2. General solution of the governing equations for a planar decagonal quasicrystal

According to the description of a n-dimensional quasicrystal as a quasiperiodic structure which is periodic in $(3+n)$-dimensional space $(1 \leqslant n \leqslant 3)$, the $(3+n)$-dimensional space can be divided into the direct sum of two orthogonal subspaces, one being three-dimensional physical or parallel space, $E_\parallel$, and the other being n-dimensional perpendicular or complementary space, $E_\perp$. Therefore, for each quasicrystal, there are two orthogonal coordinate systems, one in $E_\parallel$ and the other in $E_\perp$. In addition to the usual phonon displacements u_i and phonon strains ε_{ij} describing the local shifts of atoms in $E_\parallel$, one must introduce the phason displacements w_i and phason strains w_{ij} to describe the local rearrangements of atoms in $E_\perp$. In this section, we consider a planar decagonal quasicrystal of point group 10, which refers to a planar medium with decagonal symmetry. In this case, the generalized Hooke law is as follows [11, 17]:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl} \qquad H_{ij} = K_{ijkl}w_{kl} + R_{klij}\varepsilon_{kl} \tag{1}$$

with

$$\varepsilon_{ij} = (\partial_j u_i + \partial_i u_j)/2 \qquad w_{ij} = \partial_j w_i \tag{2}$$

where $\partial_j = \partial/\partial x_j$ (the argument x_j is always in $E_\parallel$), σ_{ij} and H_{ij} are the stresses in $E_\parallel$ and $E_\perp$, respectively, C_{ijkl} and K_{ijkl} elastic constants in the phonon and the phason field, respectively, and R_{ijkl} the phonon–phason coupling elastic constants. Moreover,

$$\begin{aligned} C_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ K_{ijkl} &= K_1\delta_{ik}\delta_{jl} + K_2(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}) \\ R_{ijkl} &= R_1(\delta_{i1} - \delta_{i2})(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &\quad + R_2[(1-\delta_{ij})\delta_{kl} + \delta_{ij}(\delta_{i1} - \delta_{i2})(\delta_{k1}\delta_{l2} - \delta_{k2}\delta_{l1})] \end{aligned} \tag{3}$$

where $i, j, k, l = 1, 2$.

Substituting (1) and (2) into the equilibrium equations

$$\partial_j\sigma_{ij} = 0 \qquad \partial_j H_{ij} = 0 \tag{4}$$

leads to the basic governing equations for a planar decagonal quasicrystal of point group 10 below:

$$C_{ijkl}\partial_j\partial_l u_k + R_{ijkl}\partial_j\partial_l w_k = 0 \tag{5a}$$

$$R_{klij}\partial_j\partial_l u_k + K_{ijkl}\partial_j\partial_l w_k = 0. \tag{5b}$$

Applying the theory of partial differential equations (e.g. see [18]), it follows from equations (5) that the displacement field can be expressed in terms of two functions φ and ψ as follows:

$$u_1 = (\mu+\lambda)\partial_1\partial_2\varphi + \mu\partial_1^2\psi + (2\mu+\lambda)\partial_2^2\psi \tag{6a}$$

$$u_2 = -[(2\mu+\lambda)\partial_1^2\varphi + \mu\partial_2^2\varphi + (\mu+\lambda)\partial_1\partial_2\psi] \tag{6b}$$

$$w_1 = -\omega\{[2R_1\partial_1\partial_2 - R_2(\partial_1^2 - \partial_2^2)]\varphi + [R_1(\partial_1^2 - \partial_2^2) + 2R_2\partial_1\partial_2]\psi\} \tag{6c}$$

$$w_2 = \omega\{[R_1(\partial_1^2 - \partial_2^2) + 2R_2\partial_1\partial_2]\varphi - [2R_1\partial_1\partial_2 - R_2(\partial_1^2 - \partial_2^2)]\psi\} \tag{6d}$$

where

$$\omega = \mu(2\mu+\lambda)/R^2 \qquad R^2 = R_1^2 + R_2^2. \tag{7}$$

Inserting expressions (6) into equations (5) reveals that equations (5*a*) are identically satisfied and that equations (5*b*) become

$$(2\mu+\lambda)c_2\partial_1\Lambda_1\varphi+\mu c_1\partial_2\Lambda_2\varphi+(2\mu+\lambda)c_2\partial_2\Lambda_1\psi-\mu c_1\partial_1\Lambda_2\psi=0 \qquad (8a)$$

$$(2\mu+\lambda)c_2\partial_1\Lambda_2\varphi-\mu c_1\partial_2\Lambda_1\varphi+(2\mu+\lambda)c_2\partial_2\Lambda_2\psi+\mu c_1\partial_1\Lambda_1\psi=0 \qquad (8b)$$

where

$$\begin{aligned}&\Lambda_1=R_1\partial_2(3\partial_1^2-\partial_2^2)+R_2\partial_1(3\partial_2^2-\partial_1^2) &&\Lambda_2=R_1\partial_1(3\partial_2^2-\partial_1^2)-R_2\partial_2(3\partial_1^2-\partial_2^2)\\ &c_1=(2\mu+\lambda)K_1-R^2 &&c_2=\mu K_1-R^2.\end{aligned} \qquad (9)$$

A further simplification of (8) is achieved if we now choose a new function F, which is called the displacement function, such that

$$\varphi=-(2\mu+\lambda)c_2R\partial_2\Lambda_1F+\mu c_1R\partial_1\Lambda_2F \qquad (10a)$$

$$\psi=(2\mu+\lambda)c_2R\partial_1\Lambda_1F+\mu c_1R\partial_2\Lambda_2F. \qquad (10b)$$

In this case, it is easily seen that (8*a*) is automatically satisfied and (8*b*) leads to

$$\Delta^4F=0 \qquad (11)$$

where $\Delta=\partial_1^2+\partial_2^2$. Thus, the system of equations (5) is reduced to a single equation (11) for F. Substituting (10) into (6) yields the displacement and stress fields:

$$u_1=R[c_2\partial_1\Lambda_1+c_1\partial_2\Lambda_2]\Delta F \qquad (12a)$$

$$u_2=R[c_2\partial_2\Lambda_1-c_1\partial_1\Lambda_2]\Delta F \qquad (12b)$$

$$w_1=-c_0\Lambda_1\Lambda_2F \qquad (12c)$$

$$w_2=-R^{-1}[c_2(2\mu+\lambda)\Lambda_1^2+c_1\mu\Lambda_2^2]F \qquad (12d)$$

$$\sigma_{11}=2c_0c_2\partial_2^2\Lambda_1\Delta F \qquad (13a)$$

$$\sigma_{22}=2c_0c_2\partial_1^2\Lambda_1\Delta F \qquad (13b)$$

$$\sigma_{12}=\sigma_{21}=-2c_0c_2\partial_1\partial_2\Lambda_1\Delta F \qquad (13c)$$

$$H_{11}=-c_1c_2R\partial_2\Delta^3F+R^{-1}K_0\partial_2[c_2(2\mu+\lambda)\Lambda_1^2+c_1\mu\Lambda_2^2]F \qquad (13d)$$

$$H_{12}=c_1c_2R\partial_1\Delta^3F-R^{-1}K_0\partial_1[c_2(2\mu+\lambda)\Lambda_1^2+c_1\mu\Lambda_2^2]F \qquad (13e)$$

$$H_{21}=-c_1c_2R\partial_1\Delta^3F-c_0K_0\partial_2\Lambda_1\Lambda_2F \qquad (13f)$$

$$H_{22}=-c_1c_2R\partial_2\Delta^3F+c_0K_0\partial_1\Lambda_1\Lambda_2F \qquad (13g)$$

where

$$c_0=R(\mu+\lambda) \qquad K_0=K_1-K_2. \qquad (14)$$

Expressions (12) and (13) give a general solution in terms of F for a planar decagonal quasicrystal. Once F satisfying equation (11) is determined from given boundary conditions, the displacement and stress fields will readily be calculated from (12) and (13), respectively.

3. Solution to a dislocation in a planar decagonal quasicrystal

Utilizing the formulae of a general solution suggested above and the Fourier transform technique, the analytical solution for a dislocation in a planar decagonal quasicrystal of point group 10 can be obtained. Consider a dislocation with the core at the origin: the Burgers vector is denoted as $\boldsymbol{b}=b^{\parallel}\oplus b^{\perp}=(b_1^{\parallel},b_2^{\parallel},b_1^{\perp},b_2^{\perp})$, where

$$\oint \mathrm{d}u_j=b_j^{\parallel} \qquad \oint \mathrm{d}w_j=b_j^{\perp} \qquad (15)$$

in which the integrals in (15) should be taken along the Burgers circuit surrounding the dislocation core in $E_\parallel$ [19]. Here we calculate only the elastic field for a typical problem, which corresponds to $b_1^\parallel \neq 0$, $b_1^\perp \neq 0$, $b_2^\parallel = b_2^\perp = 0$. For brevity, we decompose this problem into two separate cases, one for $b_1^\perp = 0$ and the other for $b_1^\parallel = 0$.

First, for the former, it is concluded that σ_{22}, w_1, w_2 vanish on the x_1 axis due to symmetry and so the boundary conditions in the upper half-plane can be summarized below if denoting $x_1 = x$, $x_2 = y$

$$\sigma_{22}(x, 0) = 0 \tag{16a}$$

$$w_1(x, 0) = w_2(x, 0) = 0 \tag{16b, c}$$

$$\oint \mathrm{d}u_1 = b_1^\parallel \qquad \oint \mathrm{d}u_2 = 0 \tag{16d, e}$$

$$\sigma_{ij}(x, y) \to 0 \qquad H_{ij}(x, y) \to 0 \qquad \sqrt{x^2 + y^2} \to \infty. \tag{17}$$

Performing the Fourier transform of equation (11) with respect to x yields an ordinary differential equation, which, on account of the regularity conditions at infinity (17), possesses a solution of the form

$$\hat{F} = (4\xi^4 R^2)^{-1} XY\, \mathrm{e}^{-|\xi|y} \tag{18}$$

where $X = (A, B, C, D)$, $Y = (1, y, y^2, y^3)^T$ and $\hat{F}$ is the Fourier transform of F, defined by

$$\hat{F} = \int_{-\infty}^{+\infty} F(x, y)\, \mathrm{e}^{\mathrm{i}\xi x}\, \mathrm{d}x \tag{19}$$

where A, B, C and D are arbitrary functions of ξ to be determined from boundary conditions (16), the superscript T denotes the transposition of a matrix and the factor $(4\xi^4 R^2)^{-1}$ in (18) is introduced for convenience.

Inserting (18) into the Fourier transformed forms of (12) results in

$$\hat{u}_1 = \mathrm{i}\xi^{-1}\tilde{R}_0 X[2n\xi^2 Y' + (m - 5n)|\xi|Y'' - (2m - 5n)Y''']\, \mathrm{e}^{-|\xi|y} \tag{20a}$$

$$\hat{u}_2 = |\xi|^{-1}\tilde{R}_0 X[2n\xi^2 Y' - (m + 5n)|\xi|Y'' + (2m + 5n)Y''']\, \mathrm{e}^{-|\xi|y} \tag{20b}$$

$$\hat{w}_1 = \mathrm{i}c_0\xi^{-1}\tilde{R}_0^2 X[4|\xi|^3 Y - 12\xi^2 Y' + 15|\xi|Y'' - 10Y''']\, \mathrm{e}^{-|\xi|y} \tag{20c}$$

$$\hat{w}_2 = c_0|\xi|^{-1}\tilde{R}_0^2 X[4|\xi|^3 Y - 12\xi^2 Y' + 15|\xi|Y'' - (10 + e_0 R_0^2)Y''']\, \mathrm{e}^{-|\xi|y} \tag{20d}$$

where

$$m = c_2 + c_1 \qquad n = c_2 - c_1 \qquad e_0 = -[\mu c_1 + (2\mu + \lambda)c_2]/(Rc_0) \tag{21}$$

$$R_0 = (R_1 + \mathrm{i}R_2\,\mathrm{sgn}\,\xi)/R \qquad \tilde{R}_0 = (R_1 - \mathrm{i}R_2\,\mathrm{sgn}\,\xi)/R. \tag{22}$$

Similarly, it follows from (13) that

$$\hat{\sigma}_{11} = 2c_0c_2R^{-1}\tilde{R}_0 X(-2\xi^2 Y' + 8|\xi|Y'' - 13Y''')\, \mathrm{e}^{-|\xi|y} \tag{23a}$$

$$\hat{\sigma}_{22} = 2c_0c_2R^{-1}\tilde{R}_0 X(2\xi^2 Y' - 4|\xi|Y'' + 3Y''')\, \mathrm{e}^{-|\xi|y} \tag{23b}$$

$$\hat{\sigma}_{12} = \hat{\sigma}_{21} = \mathrm{i}2c_0c_2R^{-1}\tilde{R}_0(\mathrm{sgn}\,\xi)X(2\xi^2 Y' - 6|\xi|Y'' + 7Y''')\, \mathrm{e}^{-|\xi|y} \tag{23c}$$

$$\hat{H}_{11} = c_0K_0\tilde{R}_0^2 X[4|\xi|^3 Y - 16\xi^2 Y' + 27|\xi|Y'' - (25 + e_2R_0^2)Y''']\, \mathrm{e}^{-|\xi|y} \tag{23d}$$

$$\hat{H}_{22} = c_0K_0\tilde{R}_0^2 X[-4|\xi|^3 Y + 12\xi^2 Y' - 15|\xi|Y'' + (10 - e_1R_0^2)Y''']\, \mathrm{e}^{-|\xi|y} \tag{23e}$$

$$\hat{H}_{12} = \mathrm{i}c_0K_0\tilde{R}_0^2(\mathrm{sgn}\,\xi)X[-4|\xi|^3 Y + 12\xi^2 Y' - 15|\xi|Y'' + (10 + e_2R_0^2)Y''']\, \mathrm{e}^{-|\xi|y} \tag{23f}$$

$$\hat{H}_{21} = \mathrm{i}c_0K_0\tilde{R}_0^2(\mathrm{sgn}\,\xi)X[-4|\xi|^3 Y + 16\xi^2 Y' - 27|\xi|Y'' + (25 - e_1R_0^2)Y''']\, \mathrm{e}^{-|\xi|y} \tag{23g}$$

where sgn ξ is equal to 1 as $\xi > 0$ or -1 as $\xi < 0$,

$$e_1 = \frac{2c_1c_2}{c_0K_0R} \qquad e_2 = \frac{c_1c_2}{c_0K_0R}\left(\frac{c_1'}{c_1} + \frac{c_2'}{c_2}\right)$$
$$c_1' = (2\mu+\lambda)K_2 - R^2 \qquad c_2' = \mu K_2 - R^2. \tag{24}$$

A comparison of the boundary conditions (16*a*–*c*) with (20*c*, *d*) and (22*b*) for $y = 0$ yields

$$4A_1 = 9C_1 \qquad B_1 = 2C_1 \qquad D_1 = 0 \tag{25}$$

where $A_1 = A|\xi|^3$, $B_1 = B\xi^2$, $C_1 = 2C|\xi|$, $D_1 = 6D$ and C_1 is an arbitrary function of ξ to be determined from the remaining boundary conditions (16*d*, *e*).

Upon substitution of the above results into (20*a*, *b*), after some algebra, we have

$$\oint \mathrm{d}u_1 = -4c_1R^{-1}(R_1 \operatorname{Re} C_1 + R_2 \operatorname{sgn}\xi \operatorname{Im} C_1) \tag{26a}$$

$$\oint \mathrm{d}u_2 = 4c_1R^{-1}(R_2 \operatorname{Re} C_1 - R_1 \operatorname{sgn}\xi \operatorname{Im} C_1) \tag{26b}$$

where Re and Im denote the real and the imaginary parts of a complex variable, respectively.

Comparing (16*d*, *e*) with (26) yields

$$C_1 = -R_0b_1^{\|}/(4c_1). \tag{27}$$

Therefore, substituting (27) in conjunction with (25) into (20) and making use of the inverse Fourier transform results in

$$u_1 = \frac{b_1^{\|}}{2\pi}\left[\tan^{-1}\left(\frac{y}{x}\right) + \frac{c_1-c_2}{c_1}\frac{xy}{r^2}\right] \tag{28a}$$

$$u_2 = \frac{b_1^{\|}}{2\pi}\left[-\ln\frac{r}{a} + \frac{c_1-c_2}{c_1}\left(\ln\frac{r}{a} + \frac{y^2}{r^2}\right)\right] \tag{28b}$$

$$w_1 = \frac{c_0b_1^{\|}}{2\pi c_1}\left[\frac{R_1}{R}\frac{2x^3y}{r^4} + \frac{R_2}{R}\frac{y^2(3x^2+y^2)}{r^4}\right] \tag{28c}$$

$$w_2 = \frac{c_0b_1^{\|}}{2\pi c_1}\left[\frac{R_1}{R}\frac{y^2(3x^2+y^2)}{r^4} + \frac{R_2}{R}\frac{2x^3y}{r^4}\right] \tag{28d}$$

where $r = \sqrt{x^2+y^2}$ and a, the radius of the dislocation core, makes the log term dimensionless.

From (23), by using (25) and (27) we obtain

$$\sigma_{11} = -\frac{c_0c_2b_1^{\|}}{\pi c_1R}\frac{y(3x^2+y^2)}{r^4}$$

$$\sigma_{22} = \frac{c_0c_2b_1^{\|}}{\pi c_1R}\frac{y(x^2-y^2)}{r^4}$$

$$\sigma_{12} = \sigma_{21} = \frac{c_0c_2b_1^{\|}}{\pi c_1R}\frac{x(x^2-y^2)}{r^4}$$

$$H_{11} = -\frac{c_0K_0b_1^{\|}}{\pi c_1}\left[\frac{R_1}{R}\frac{x^2y(3x^2-y^2)}{r^6} + \frac{R_2}{R}\frac{x^3(3y^2-x^2)}{r^6}\right]$$

$$H_{22} = -\frac{c_0K_0b_1^{\|}}{\pi c_1}\left[\frac{R_1}{R}\frac{x^2y(3y^2-x^2)}{r^6} - \frac{R_2}{R}\frac{xy^2(3x^2-y^2)}{r^6}\right]$$

$$H_{12} = -\frac{c_0K_0b_1^{\|}}{\pi c_1}\left[\frac{R_1}{R}\frac{xy^2(3x^2-y^2)}{r^6} + \frac{R_2}{R}\frac{x^2y(3y^2-x^2)}{r^6}\right]$$

$$H_{21} = -\frac{c_0 K_0 b_1^{\|}}{\pi c_1}\left[-\frac{R_1}{R}\frac{x^3(3y^2-x^2)}{r^6} + \frac{R_2}{R}\frac{x^2y(3x^2-y^2)}{r^6}\right]. \tag{29}$$

Second, for the latter, i.e. for the case of $b_1^{\|} = 0$, an analogous procedure, as just solved above, reduces to

$$\begin{aligned}
u_1 &= \frac{c_1 b_1^{\perp}}{\pi c_0 e_1}\left\{\frac{R_1}{R}\left[\frac{xy}{r^2} - \frac{c_1-c_2}{c_1}\frac{2xy^3}{r^4}\right] + \frac{R_2}{R}\left[\frac{y^2}{r^2} + \frac{c_1-c_2}{c_1}\frac{y^2(x^2-y^2)}{r^4}\right]\right\}\\
u_2 &= \frac{c_1 b_1^{\perp}}{\pi c_0 e_1}\left\{-\frac{R_1}{R}\left[\frac{y^2}{r^2} - \frac{c_1-c_2}{c_1}\frac{y^2(x^2-y^2)}{r^4}\right] + \frac{R_2}{R}\left[\frac{xy}{r^2} + \frac{c_1-c_2}{c_1}\frac{2xy^3}{r^4}\right]\right\}\\
w_1 &= \frac{b_1^{\perp}}{2\pi}\left[\tan^{-1}\frac{y}{x} + \frac{(R_1^2-R_2^2)}{e_1R^2}\frac{xy(3x^2-y^2)(3y^2-x^2)}{3r^6} + \frac{2R_1R_2}{e_1R^2}\frac{y^2(3x^2-y^2)^2}{3r^6}\right]\\
w_2 &= \frac{b_1^{\perp}}{2\pi e_1}\left[e_2\ln\frac{r}{a} + \frac{(R_1^2-R_2^2)}{R^2}\frac{y^2(3x^2-y^2)^2}{3r^6} - \frac{2R_1R_2}{R^2}\frac{xy(3x^2-y^2)(3y^2-x^2)}{3r^6}\right]\\
\sigma_{11} &= -\frac{2c_2b_1^{\perp}}{\pi e_1 R}\left[\frac{R_1}{R}\frac{x^2y(3x^2-y^2)}{r^6} + \frac{R_2}{R}\frac{x^3(3y^2-x^2)}{r^6}\right]\\
\sigma_{22} &= -\frac{2c_2b_1^{\perp}}{\pi e_1 R}\left[\frac{R_1}{R}\frac{y^3(3x^2-y^2)}{r^6} + \frac{R_2}{R}\frac{xy^2(3y^2-x^2)}{r^6}\right]\\
\sigma_{12} &= \sigma_{21} = -\frac{2c_2b_1^{\perp}}{\pi e_1 R}\left[\frac{R_1}{R}\frac{xy^2(3x^2-y^2)}{r^6} + \frac{R_2}{R}\frac{x^2y(3y^2-x^2)}{r^6}\right]\\
H_{11} &= \frac{K_0 b_1^{\perp}}{2\pi e_1}\left\{-(e_1+e_2)\frac{y}{r^2} + x\left[\frac{(R_1^2-R_2^2)}{R^2}h_{21}(x,y) - \frac{2R_1R_2}{R^2}h_{22}(x,y)\right]\right\}\\
H_{12} &= \frac{K_0 b_1^{\perp}}{2\pi e_1}\left\{(e_1+e_2)\frac{x}{r^2} + y\left[\frac{(R_1^2-R_2^2)}{R^2}h_{21}(x,y) - \frac{2R_1R_2}{R^2}h_{22}(x,y)\right]\right\}\\
H_{21} &= -\frac{K_0 b_1^{\perp} x}{2\pi e_1}\left[\frac{(R_1^2-R_2^2)}{R^2}h_{22}(x,y) + \frac{2R_1R_2}{R^2}h_{21}(x,y)\right]\\
H_{22} &= -\frac{K_0 b_1^{\perp} y}{2\pi e_1}\left[\frac{(R_1^2-R_2^2)}{R^2}h_{22}(x,y) + \frac{2R_1R_2}{R^2}h_{21}(x,y)\right]
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
h_{21}(x,y) &= \frac{2xy(3x^2-y^2)(3y^2-x^2)}{r^8}\\
h_{22}(x,y) &= \frac{2(x^2-y^2)}{r^4} + \frac{(x^2-y^2)(3x^2-y^2)(3y^2-x^2)}{r^8}.
\end{aligned} \tag{31}$$

Consequently, the sum of the expressions in (28), (29) and (30) for the corresponding variables will give the analytical expressions, denoted as $u_j^{(1)}$, $w_j^{(1)}$, $\sigma_{ij}^{(1)}$ and $H_{ij}^{(1)}$, for the elastic field induced by a dislocation $(b_1^{\|}, 0, b_1^{\perp}, 0)$ in a planar decagonal quasicrystal of point group 10.

For another typical problem, in which the Burgers vector of a dislocation is denoted by $(0, b_2^{\|}, 0, b_2^{\perp})$, an entirely similar consideration will yield similar results, which are omitted here. Alternatively, the expressions, denoted as $u_j^{(2)}$, $w_j^{(2)}$, $\sigma_{ij}^{(2)}$ and $H_{ij}^{(2)}$, for the elastic field for $(0, b_2^{\|}, 0, b_2^{\perp})$ can also be determined from those for $(b_1^{\|}, 0, b_1^{\perp}, 0)$ by rotating the coordinate system [16], i.e. only making the substitutions $x \to y$, $y \to -x$, $b_1^{\|} \to b_2^{\|}$, $b_1^{\perp} \to -b_2^{\perp}$, $u_1^{(1)} \to u_2^{(2)}$, $u_2^{(1)} \to -u_1^{(2)}$, $w_1^{(1)} \to -w_2^{(2)}$ and $w_2^{(1)} \to w_1^{(2)}$. Therefore, the explicit analytical expressions for the elastic field for a dislocation $(b_1^{\|}, b_2^{\|}, b_1^{\perp}, b_2^{\perp})$ in a planar decagonal quasicrystal of point group 10 can be obtained by superposition of the corresponding expressions for the elastic fields for $(b_1^{\parallel}, 0, b_1^{\perp}, 0)$ and $(0, b_2^{\parallel}, 0, b_2^{\perp})$, namely

$$\begin{aligned}
&u_j = u_j^{(1)} + u_j^{(2)} \qquad w_j = w_j^{(1)} + w_j^{(2)} \qquad \sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \\
&H_{ij} = H_{ij}^{(1)} + H_{ij}^{(2)} \qquad i, j = 1, 2.
\end{aligned} \tag{32}$$

4. A straight dislocation in a decagonal quasicrystal

As we know, any realistic quasicrystal is a spatial solid, not a planar medium. Therefore, in this section, we consider a straight dislocation line in a decagonal quasicrystal with Laue class $10/m$, or with point groups 10, $\overline{10}$ and $10/m$; here a decagonal quasicrystal refers to a three-dimensional solid periodically stacked in a two-dimensional quasiperiodic structure with decagonal symmetry. Then, the only discrepancy between a decagonal quasicrystal and a planar decagonal quasicrystal is that there is an additional phonon displacement, u_3, along the periodic direction for the former. For convenience, the generalized Hooke law is still expressed by (5) where $i, j = 1, 2, 3$, $w_3 \equiv 0$ and the elastic constant matrices $[C]$, $[K]$ and $[R]$ can be obtained by group representation theory [17, 20], i.e. the expressions for the elastic constants K_{ijkl} and R_{ijkl} take the same forms as the last two in (3), while for the elastic constants C_{ijkl},

$$\begin{aligned}
&C_{1111} = C_{2222} = C_{11} \qquad C_{3333} = C_{33} \qquad C_{1133} = C_{2233} = C_{13} \\
&C_{2323} = C_{3131} = C_{44} \qquad C_{1122} = C_{12} \qquad 2C_{1212} = C_{1111} - C_{1122} = C_{66}.
\end{aligned}$$

In this case, there are ten independent elastic constants, five in the phonon field, three in the phason and two associated with the phonon–phason coupling. For a straight dislocation line in a decagonal quasicrystal, the explicit expressions for the elastic field, in general, are quite difficult to derive, mainly due to the complication of solving a tenth-order linear algebraic equation associated with the governing equations. Of course, the theoretically formal solution can be obtained by means of the generalized Lekhnitskii–Eshelby–Stroh method [21]. However, the explicit analytical expressions for the elastic field can be obtained for the special case where a straight dislocation line is parallel to the periodic direction, the x_3 axis. In this case, it is evident that all the variables are independent of x_3, namely $\partial_3 = 0$. It further indicates that the governing equations (5) where $i, j = 1, 2, 3$ and $w_3 \equiv 0$ for a decagonal quasicrystal can be decoupled into the variables in the quasiperiodic plane and the phonon displacement u_3 along the periodic direction. Moreover, for the quasiperiodic plane the variables satisfy the same equations as (5) with $i, j = 1, 2$ and only taking C_{11} and C_{66} instead of $2\mu + \lambda$ and μ, respectively, and for u_3 it satisfies

$$\Delta u_3 = 0. \tag{33}$$

Consider a straight dislocation line parallel to the periodic direction in a decagonal quasicrystal with the Burgers vector $\boldsymbol{b} = b^{\parallel} \oplus b^{\perp} = (b_1^{\parallel}, b_2^{\parallel}, b_3^{\parallel}, b_1^{\perp}, b_2^{\perp})$. Based on the above analysis, the elastic field can be obtained as follows:

$$\begin{aligned}
&u_j = u_j^{(1)} + u_j^{(2)} \qquad w_j = w_j^{(1)} + w_j^{(2)} \qquad j = 1, 2 \\
&u_3 = \frac{b_3^{\parallel}}{2\pi} \tan^{-1} \frac{y}{x} \\
&\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \qquad H_{ij} = H_{ij}^{(1)} + H_{ij}^{(2)} \qquad i, j = 1, 2 \\
&\sigma_{13} = \sigma_{31} = -\frac{C_{44} b_3^{\parallel}}{2\pi} \frac{y}{r^2} \qquad \sigma_{23} = \sigma_{32} = \frac{C_{44} b_3^{\parallel}}{2\pi} \frac{x}{r^2}
\end{aligned} \tag{34}$$

where u_j, w_j, σ_{ij} and H_{ij} $(i,\ j = 1, 2)$ are those given by (32), which corresponds to the elastic field for $(b_1^{\|}, b_2^{\|}, 0, b_1^{\perp}, b_2^{\perp})$, only taking into account C_{11} and C_{66} instead of $2\mu + \lambda$ and μ, respectively, in all the expressions in the preceding section.

The energy per unit length on a straight dislocation line can be evaluated by the integral

$$W = \frac{1}{2}\int (\sigma_{ij}\varepsilon_{ij} + H_{ij}w_{ij})\,\mathrm{d}S \tag{35}$$

where the integral should be taken over a circular annulus $a \leqslant r \leqslant R_0$, a the radius of the dislocation core. After some manipulation, the final result of the energy per unit length on a straight dislocation with the Burgers vector $(b_1^{\|}, 0, b_3^{\|}, b_1^{\perp}, 0)$ is

$$W = \left[\frac{2c_0c_2(b_1^{\|})^2}{c_1R} + \frac{K_0(e_1+e_2)(b_1^{\perp})^2}{e_1} + C_{44}(b_3^{\|})^2\right]\frac{1}{4\pi}\ln\frac{R_0}{a}. \tag{36}$$

If $b_1^{\perp} = \zeta b_1^{\|}$, $b_1^{\|} = b^{\|}\sin\varphi$ and $b_3^{\|} = b^{\|}\cos\varphi$, where ζ is a constant, $b^{\|}$ is the magnitude of the phonon part $(b_1^{\|}, 0, b_3^{\|})$ in $E_{\|}$ of a dislocation $\boldsymbol{b}$ and φ is the angle in $E_{\|}$ between the vector of the phonon part $(b_1^{\|}, 0, b_3^{\|})$ in $E_{\|}$ of a dislocation $\boldsymbol{b} = (b_1^{\|}, 0, b_3^{\|}, b_1^{\perp}, 0)$ and the dislocation line, we have

$$W = \left[\frac{2c_0c_2\sin^2\varphi}{c_1R} + \frac{K_0(e_1+e_2)\zeta^2\sin^2\varphi}{e_1} + C_{44}\cos^2\varphi\right]\frac{(b^{\|})^2}{4\pi}\ln\frac{R_0}{a}. \tag{37}$$

5. Discussion and conclusion

For a planar decagonal quasicrystal of point group $10mm$ with the presence of a dislocation, there are only five independent elastic constants [11], i.e. it corresponds to the special case of a planar decagonal quasicrystal of point group 10: $R_2 = 0$. Therefore, the present solution for the point group 10, if setting $R_2 = 0$ and so $R = R_1$, reduces to the result obtained in [15] and [16].

Similarly, for a straight dislocation line parallel to the periodic direction in a decagonal quasicrystal with Laue class $10/mmm$, or with point groups $10mm$, 1022, $\overline{10}m2$ and $10/mmm$, for which there are nine independent elastic constants, the dislocation-induced elastic field can be obtained from the limitation $R_2 = 0$ of the results in a decagonal quasicrystal with Laue class $10/m$, or with point groups 10, $\overline{10}$ and $10/m$, which is the same as the results given in [11].

On the other hand, when the displacement function F takes some simple polynomials such as a_0x^7, a_1x^6y, b_0x^8, b_1x^7y etc, for example, the results given by (12) and (13) are several simple elastic solutions suitable for treating a decagonal quasicrystal rectangular plate under the action of uniform or linear tension or pressure. However, these simple elastic solutions seem to be not easy to obtain by other methods.

By the way, one can apply the expression for the energy on a dislocation given by (36) to analyse the dissociation of a dislocation in a decagonal quasicrystal. Unlike the method mentioned above, some other methods have been developed to study the dissociation of a dislocation in quasicrystals in [22] and [23].

The present solution can be used as a fundamental solution for a dislocation in a decagonal quasicrystal. Therefore, many elasticity problems in a decagonal quasicrystal can be directly solved with the aid of this fundamental solution by superposition.

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